Solution Key Module 1

Q1) Find the extremals of $\int_{x_1}^{x_2}(1+x^2y')y'\,dx$?

Solution :

$$\because \; F = (1+x^2y')\,y'$$

here f does not contains y explicitly

so by formula

$$F = y' + x^2y'^2$$

$$\therefore \; \frac{\partial F}{\partial y'} = 1 + 2x^2y'$$

$$\because \; \frac{\partial F}{\partial y'} = c \qquad \text{(formula)}$$

$$1 + 2x^2y' = c'$$

$$\therefore \; 2x^2y' = c' - 1 = c_1$$

$$\therefore \; y' = \frac{c_1}{2x^2}$$

$$y' = \frac{c}{x^2}$$

Integrating both the sides

$$\int y'\,dy = \int \frac{c}{x^2}\,dx$$

$$y = -\frac{c}{x} + c_2$$

Changing the constants

$$y = \frac{c_1}{x} + c_2$$

The extremals of $\int_{x_1}^{x_2}(1+x^2y')y'\,dx$ is $y = \frac{c_1}{x} + c_2$

Author : Khyati Baria

Q2) Find the extremal of the function $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dy$ with $y(0) = 0$; $y\left(\frac{\pi}{2}\right) = 0$

Solution :

$$\because F = y'^2 - y^2 + 2xy$$

$$\frac{\partial F}{\partial y} = -2y + 2x$$

and $\quad \because \dfrac{\partial F}{\partial y} = 2y'$

Substituting values in Euler Equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$-2y + 2x - \frac{d}{dx}(2y') = 0$$

$$\therefore \frac{d^2y}{dx^2} + y = x$$

The A.E is $D^2 + 1 = 0$

$$\therefore (D+1)(D-1) = 0$$

$$\therefore D = +i ; -i$$

$$CF = y = c_1 \cos x + c_2 \sin x$$

$$PI = \frac{1}{D^2+1} \cdot x$$

$$= (D^2+1)^{-1} x$$

$$= (1 - D^2 + D^4 - \dots)x \quad = x$$

The complete solution is $y = c_1 \cos x + c_2 \sin x + x$

when $x = 0$; $y = 0$

$$0 = c_1$$

when $x = \frac{\pi}{2}$; $y = 0$

$$0 = c_2 + \frac{\pi}{2}$$

$\therefore C_2 = -\frac{\pi}{2}$

$\therefore y = x - \frac{\pi}{2}\sin x$

The extremal of function is $y = x - \frac{\pi}{2}\sin x$.

Q3) Find the extremal of $\int_0^1 (xy + y^2 - 2y^2y')\,dx$

Solution :

$\because F = xy + y^2 - 2y^2y'$

$\frac{\partial F}{\partial y} = x + 2y - 4yy'$

and $\frac{\partial F}{\partial y'} = -2y^2$

Substituting these values in Euler's Equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$\therefore x + 2y - 4yy' - \frac{d}{dx}(-2y^2) = 0$

$\therefore x + 2y - 4yy' + 4yy' = 0$

$\therefore x + 2y = 0$

$\therefore y = -\frac{x}{2}$

$\therefore$ The extremal of function is $y = \frac{-x}{2}$

Q4) Find the extremal of $\int_{x_0}^{x_1} (2xy - y''^2)\,dx$

Solution :

$\because F = 2xy - y''^2$

$\frac{\partial F}{\partial y} = 2x$

$$\frac{\partial f}{\partial y'} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y''} = -2y''$$

$\because$ $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$ then the equation is extremum.

$$\therefore \quad 2x - 0 + \frac{d^2}{dx^2}(-2y'') = 0$$

$$\therefore \quad 2x - \frac{2d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore \quad 2x - 2\frac{d^4y}{dx^4} = 0$$

$$\therefore \quad \frac{d^4y}{dx^4} = x$$

The above equation is linear differential equation of foueth order.

Its A.E is $D^4 = 0$

$D = 0, 0, 0, 0$

$$PI = \frac{1}{D^4} x$$

$$\therefore y = \frac{1}{D^3}\int x\,dx = \frac{1}{D^3} \cdot \frac{x^2}{2}$$

$$\therefore \quad = \frac{1}{D^2}\int \frac{x^2}{2}dx \qquad = \frac{1}{D^2} \cdot \frac{x^3}{6}$$

$$\therefore \quad = \frac{1}{D}\int \frac{x^3}{6}dx \qquad = \frac{1}{D}\frac{x^4}{24}$$

$$\therefore \quad = \int \frac{x^4}{24}dx$$

$$y = \frac{x^5}{120}$$

Hence $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{x^5}{120}$ is the solution.

Q5). Using Rayleigh - Ritz Method solve the boundary value problem $I = \int_0^1 \left(xy + \frac{1}{2} y'^2 \right) dx$; $0 \le x \le 1$ given $y(0) = 0$ and $y(1) = 0$?

Solution : $F = xy + \frac{1}{2} y'^2$

Assuming Trial solution

$$\bar{y}(x) = c_0 + c_1 x + c_2 x^2$$

$\bar{y}(0) = 0$

$\therefore c_0 = 0$

$\bar{y}(1) = 0$

$\therefore c_1 + c_2 = 0$

$\therefore c_2 = -c_1$

$$\bar{y}(x) = c_1 x - c_1 x^2 = c_1 (x - x^2)$$

$$\bar{y}'(x) = c_1 (1 - 2x)$$

Putting values in $I = \int_0^1 \left(xy + \frac{1}{2} y'^2 \right) dx$

$$I = \int_0^1 \left\{ x \left[c_1 (x - x^2) \right] + \frac{1}{2} c_1^2 (1 - 2x)^2 \right\} dx$$

$$= \int_0^1 \left[c_1 (x^2 - x^3) + \frac{1}{2} c_1^2 (1 - 4x + 4x^2) \right] dx$$

$$= c_1 \left[\int_0^1 x^2 dx - \int_0^1 x^3 dx \right] + \frac{1}{2} c_1^2 \left[\int_0^1 (1 - 4x + 4x^2) dx \right]$$

$$= G\left\{\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1\right\} + \frac{1}{2}G^2\left\{(x)_0^1 - 4\left(\frac{x^2}{2}\right)_0^1 + 4\left(\frac{x^3}{3}\right)_0^1\right\}$$

$$= G\left\{\frac{1}{3} - \frac{1}{4}\right\} + \frac{1}{2}G^2\left\{1 - 2 + \frac{4}{3}\right\}$$

$$\therefore \; = G\left(\frac{1}{12}\right) + \frac{1}{2}G^2\left(\frac{1}{3}\right)$$

$$I = \frac{G}{12} + \frac{G^2}{6}$$

Stationary values are

$$\frac{dI}{dG} = 0$$

$$\therefore \; \frac{1}{12} + \frac{2G}{6} = 0$$

$$\therefore \; G = \frac{-1}{4}$$

Putting values in eqn (1)

$$\overline{y}(x) = \frac{-1}{4}(x - x^2)$$

$$\overline{y}(x) = \frac{1}{4}x(x-1)$$

Q6) Using Rayleigh - Ritz Method solve the boundary value problem

$$I = \int_0^1 \left(xy + \frac{1}{2}y'^2\right) dx \qquad 0 \le x \le 1$$

given $y(0) = 0$ and also $y(1) = 0$?

Solution : Extremise $I = \int_0^1 \left(xy + \frac{1}{2}y'^2\right) dx$

where $F = xy + \frac{1}{2}y'^2$

Assume trial solution $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$ —— ③

Given data

$$\bar{y}(0) = 0$$

$$\therefore \quad c_0 = 0 \quad \text{and} \quad \bar{y}(1) = 0$$

$$\therefore \quad c_1 + c_2 = 0$$

$$\therefore \quad c_2 = -c_1$$

$$\therefore \bar{y}(x) = c_1 x - c_1 x^2$$

$$\bar{y}(x) = c_1(x - x^2) \quad \text{——} \quad ④$$

$$\bar{y}'(x) = c_1(1-2x)$$

Putting values in eqn 1

$$I = \int_0^1 \left(xy + \frac{1}{2} y'^2 \right) dx$$

$$I = \int_0^1 \left\{ x[c_1(x-x^2)] + \frac{1}{2} c_1^2 (1-2x)^2 \right\} dx$$

$$= \int_0^1 \left[c_1(x^2 - x^3) + \frac{1}{2} c_1^2 (1 - 4x + 4x^2) \right] dx$$

$$= c_1 \left[\int_0^1 x^2 dx - \int_0^1 x^3 dx \right] + \frac{c_1^2}{2} \left[\int_0^1 1\, dx - 4\int_0^1 x\, dx + 4\int_0^1 x^2 dx \right]$$

$$= c_1 \left\{ \left[\frac{x^3}{3} \right]_0^1 - \left[\frac{x^4}{4} \right]_0^1 \right\} + \frac{c_1^2}{2} \left\{ \left[x \right]_0^1 - 4\left[\frac{x}{2} \right]_0^1 + 4\left[\frac{x^3}{3} \right]_0^1 \right\}$$

$$= c_1 \left[\frac{1}{3} - \frac{1}{4} \right] + \frac{1}{2} c_1^2 \left[1 - 2 + \frac{4}{3} \right]$$

$$\therefore \quad I = c_1 \left(\frac{1}{12} \right) + \frac{1}{2} c_1^2 \left(\frac{1}{3} \right)$$

$$\therefore I = \frac{C_1}{12} + \frac{C_1^2}{6}$$

Its Stationary Values are given by

$$\frac{dI}{dC_1} = 0$$

$$\therefore \frac{1}{12} + \frac{2C_1}{6} = 0$$

$$\therefore C_1 = \frac{-1}{4}$$

Putting this value in eqn 4

$$\overline{y(x)} = \frac{-1}{4}(x - x^2)$$

$$\overline{y(x)} = \frac{1}{4}(x-1)x$$

$$\therefore \overline{y}(x) = \frac{1}{4}x(x-1)$$

Solution Key Module 2

Q1) Verify Cauchy - Schwartz inequality for the vectors
$u = (-4, 2, 1)$ and $v = (8, -4, -2)$

Solution : $\|u\| = \sqrt{16+4+1}$ $\|v\| = \sqrt{64+16+4}$

$\|u\| = \sqrt{21}$ $\|v\| = \sqrt{84}$

$\therefore \|u\|\|v\| = \sqrt{21}\sqrt{84} = \sqrt{21}\sqrt{21}\sqrt{4} = 42$

$|u \cdot v| = |u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4|$

$= |(-4)(8) + 2(-4) + 1(-2)|$

$= |-32 - 8 - 2|$

$= 42$

Hence $\|u\|\|v\| = |u \cdot v|$

Hence Cauchy Schwartz inequality holds good for given vectors

Q2) Find a unit vector orthogonal to both $(1,1,0)$ and $(0,1,1)$

Solution : Let the required vector be (v_1, v_2, v_3)
Since its orthogonal to both u_1 & u_2

$u_1 \cdot v = (1,1,0) \cdot (v_1\ v_2\ v_3) = v_1 + v_2 + 0 = 0$

$u_2 \cdot v = (0,1,1) \cdot (v_1\ v_2\ v_3) = 0 + v_2 + v_3 = 0$

By Cramer's Rule

$$\frac{v_1}{1-0} = \frac{-v_2}{1-0} = \frac{v_3}{1-0} = k$$

$\therefore v_1 = k$; $v_2 = -k$; $v_3 = k$

Since the normal is 1

$$V_1^2 + V_2^2 + V_3^2 = 1$$

$$\therefore k^2 + k^2 + k^2 = 1$$

$$\therefore k = \pm \frac{1}{\sqrt{3}}$$

When $k = \frac{1}{\sqrt{3}}$

$$V_1 = \frac{1}{\sqrt{3}} \; ; \; V_2 = \frac{-1}{\sqrt{3}} \; ; \; V_3 = \frac{1}{\sqrt{3}}$$

When $k = \frac{-1}{\sqrt{3}}$

$$V_1 = \frac{-1}{\sqrt{3}} \; ; \; V_2 = \frac{1}{\sqrt{3}} \; ; \; V_3 = \frac{-1}{\sqrt{3}}$$

Q3) Determine whether the set of vectors of the form (a,b,c) where $b = a+c$ from a subspace of R^3 under usual addition and scalar multiplication ?

Solution : Let $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ be two vectors in R^3

Now, $V_1 + V_2 = (a_1 + a_2 , b_1 + b_2 , c_1 + c_2)$

Since $b_1 = a_1 + c_1$ and $b_2 = a_2 + c_2$

$$b_1 + b_2 = (a_1 + c_1) + (a_2 + c_2)$$

$$= a_1 + a_2 + c_1 + c_2$$

$\therefore$ $u+v$ is of form (α, β, γ) and $\beta = \alpha + \gamma$

$\therefore$ $u+v$ is in R^3

If k is a scalar then $kv_1 = (ka_1, kb_1, kc_1)$

since $b_1 = a_1 + c_1$

$$kb_1 = k(a_1 + c_1) = ka_1 + kc_1$$

$\therefore$ ku is of form (p, q, r) and $q = p + r$

$\therefore$ ku is in R^3.

Hence the set of vectors of form (a, b, c) is subspace of R^3.

Q4) Express the following as a linear combination of $V_1 = (2, 1, 4)$ $V_2 = (1, -1, 3)$ and $V_3 = (3, 2, 5)$

a) $(6, 11, 6)$?

Solution : a) $(6, 11, 6)$

Let $w = (6, 11, 6)$

Find k_1, k_2, k_3 such that

$$w = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$(6, 11, 6) = k_1(2, 1, 4) + k_2(1, -1, 3) + k_3(3, 2, 5)$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 6 \end{bmatrix}$$

$$\Rightarrow k_1 = 4$$
$$k_2 = -5$$
$$k_3 = 1$$

$$\therefore \quad w = 4v_1 - 5v_2 + (1)v_3$$

$$(6, 11, 6) = 4(2, 1, 4) - 5(1, -1, 3) + 1(3, 2, 5)$$

Q5) Express $p(x) = 7 + 8x + 9x^2$ as linear combination of
$P_1 = 2 + x + 4x^2$; $P_2 = 1 - x + 3x^2$; $P_3 = 2 + x + 5x^2$

Solution : Let $p = k_1 P_1 + k_2 P_2 + k_3 P_3$

$$7 + 8x + 9x^2 = k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(2 + x + 5x^2)$$

$$= (2k_1 + k_2 + 2k_3) + (k_1 - k_2 + k_3)x + (4k_1 + 3k_2 + 5k_3)x^2$$

solve equations

$$2k_1 + k_2 + 2k_3 = 7 \quad \text{—— } 1$$

$$k_1 - k_2 + k_3 = 8 \quad \text{—— } 2$$

$$4k_1 + 3k_2 + 5k_3 = 9 \quad \text{—— } 3$$

taking eqn 1^{st} & 2

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 9 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 2R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ -5 \end{bmatrix}$$

By R_{23}

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -9 \end{bmatrix}$$

$\therefore$ $k_1 - k_2 + k_3 = 8$; $k_2 + k_3 = -5$

$$3k_2 = -9$$

$$\therefore k_2 = -3$$

$$\therefore K_3 = -2$$

$$K_1 = 8 + K_2 - K_3$$

$$\therefore K_1 = 7$$

Hence

$$7 + 8x + 9x^2 = 7(2 + x + 4x^2) - 3(1 - x + 3x^2) - 2(2 + x + 5x^2)$$

Q6) Let R^4 have the Euclidean inner product. Express $w = (-1, 2, 6, 0)$ in the form $w = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^\perp$ where W is spanned by $v_1 = (-1, 0, 1, 2)$ and $v_2 = (0, 1, 0, 1)$?

Solution: The project of w on W

$$\text{Proj}_W w = \langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2$$

$$\therefore \text{Proj}_W w = [(-1,2,6,0) \cdot (-1,0,1,2)] v_1 + [(-1,2,6,0) \cdot (0,1,0,1)] v_2$$

$$= 7v_1 + 2v_2$$

$$= 7(-1,0,1,2) + 2(0,1,0,1)$$

$$\therefore w_1 = (-7, 2, 7, 16)$$

The component of w orthogonal to W is

$$\text{proj}_{W^\perp} w = w - \text{proj}_W w$$

$$= (-1,2,6,0) - (-7,2,7,16)$$

$$\therefore w_2 = (6, 0, -1, -16)$$

$\therefore w = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^\perp$

$$w = (-7, 2, 7, 16) + (6, 0, -1, -16)$$

Q7). Let R^3 have the Euclidean inner product. Use Gram Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis where $u_1 = (1,1,1)$, $u_2 = (-1,1,0)$, $u_3 = (1,2,1)$

Solution: Step 1: $v_1 = u_1 = (1,1,1)$

Step 2: $v_2 = u_2 - \text{proj}\, u_2$

$$= u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$$

$\langle u_2, v_1 \rangle = -1+1-0 = 0$

$\|v\|^2 = 1+1+1 = 3$

$\therefore\ v_2 = (-1,1,0) - \frac{0}{3}(1,1,1)$

$v_2 = (-1,1,0)$

Step 3: $v_3 = u_3 - \text{proj}\, u_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} \cdot v_2$

$\langle u_3, v_1 \rangle = 1+2+1 = 4$

$\langle u_3, v_2 \rangle = -1+2+0 = 1$

$\|v_1\|^2 = 3$ as before

$\|v_2\|^2 = 1+1+0 = 2$

$\therefore\ v_3 = (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$

$v_3 = \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$

Hence, $v_1 = (1,1,1)$; $v_2 = (-1,1,0)$ $v_3 = \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$

from orthogonal basis for R^3

the norms of these vectors are

$$\|v_1\| = \sqrt{1+1+1} = \sqrt{3};$$

$$\|v_2\| = \sqrt{1+1+0} = \sqrt{2};$$

$$\|v_3\| = \sqrt{\frac{1}{36}+\frac{1}{36}+\frac{1}{9}} = \sqrt{6/36} = \frac{1}{\sqrt{6}}$$

Hence, the orthogonal basis for R^3 is

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

Q8) Let R^4 have the Euclidean inner product. Use Gram-Schmidt Process to transform the basis $\{u_1, u_2, u_3\}$ into orthogonal basis where $u_1 = \{1, 0, 1, 1\}$ $u_2 = \{-1, 0, -1, 1\}$ $u_3 = \{0, -1, 1, 1\}$

Solution : Step 1 : $v_1 = u_1 = \{1, 0, 1, 1\}$

Step 2 : $v_2 = u_2 - \text{Proj}\, u_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} \cdot v_1$

$$\langle u_2, v_1\rangle = -1 + 0 - 1 = -1$$

$$\langle u_2, v_1\rangle = -1$$

$$\|v_1\|^2 = 1 + 0 + 1 + 1 = 3$$

$$\therefore V_2 = (-1, 0, -1, 1) - \frac{-1}{3}(1, 0, 1, 1)$$

$$V_2 = \left(-\frac{2}{3}, 0, \frac{-2}{3}, \frac{4}{3}\right)$$

Step 3 : $V_3 = u_3 - \text{proj } u_3$

$$u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3 v_2\rangle}{\|v_2\|^2} \cdot v_2$$

$$\langle u_3 v_1\rangle = 0 + 0 + 1 + 1 = 2$$

$$\langle u_3, v_2\rangle = (0, -1, 1, 1) \cdot \left(\frac{-2}{3}, 0, \frac{-2}{3}, \frac{4}{3}\right)$$

$$= 0 + 0 - \frac{2}{3} + \frac{4}{3} = \frac{2}{3}$$

$$\|v_1\|^2 = 3$$

$$\|v_2\|^2 = \frac{4}{9} + 0 + \frac{4}{9} + \frac{16}{9}$$

$$\|V_2\|^2 = \frac{8}{3}$$

$$\therefore V_3 = (0, -1, 1, 1) - \frac{2}{3}(1, 0, 1, 1) - \frac{2/3}{8/3}\left(\frac{-2}{3}, 0, \frac{-2}{3}, \frac{4}{3}\right)$$

$$V_3 = (0, -1, 1, 1) - \frac{2}{3}(1, 0, 1, 1) - \frac{1}{4}\left(\frac{-2}{3}, 0, \frac{-2}{3}, \frac{4}{3}\right)$$

$$= (0, -1, 1, 1) + \left(\frac{-2}{3}, 0, \frac{-2}{3}, \frac{-2}{3}\right) + \left(\frac{1}{6}, 0, \frac{1}{6}, \frac{-1}{3}\right)$$

$$V_3 = \left(\frac{1}{2}, -1, \frac{1}{2}, 0\right)$$

Hence the orthogonal set is

$$V_1 = (1, 0, 1, 1) \quad V_2 = \left(\frac{-2}{3}, 0, \frac{-2}{3}, \frac{4}{3}\right) \quad V_3 = \left(\frac{1}{2}, -1, \frac{1}{2}, 0\right)$$