Module : 2 Matrices

Weightage (19-23 Mrks)

Topics Covered

Introduction.

Homogeneous Linear Equation.

Eigenvalues and Eigenvectors. (6mrks)

Cayley Hamilton Theorem. (6mrks)

Similarity of Matrices.

Algebraic and Geometric Multiplicity.

Diagonalisng Matrix. (6mrks)

Function of Square Matrix. (6mrks)

Monic and Minimal Polynomial. (6mrks)

Derogatory and Non Derogatory Matrix. (6mrks)

Author : Khyati Baria

Introduction:

A matrix is a collection of numbers arranged into a fixed number of rows and columns.

eg: $A = \begin{bmatrix} 1 & 8 \\ 4 & 9 \end{bmatrix}_{2\times2}$ $B = \begin{bmatrix} 1 & 6 & 5 \\ 0 & 4 & 2 \\ 9 & 0 & 3 \end{bmatrix}_{3\times3}$

Definition:

A matrix having m rows and n columns is called as a matrix of order 'm x n'.

eg: $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \dots & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{23} \dots & a_{2j} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} \dots & a_{ij} & a_{in} \\ a_{m1} & a_{m2} & a_{m3} \dots & a_{mj} & a_{mn} \end{bmatrix}_{m\times n}.$$

where 'm' ⟶ no. of rows
'n' ⟶ no. of columns

Transpose :

Transpose of a matrix is formed by swapping the rows into the columns and columns into rows.

eg: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Transpose of a matrix A is represented by A^T

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$ $B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}_{3\times3}$

Inverse of a Matrix:

Inverse of a matrix is represented by A^{-1}.

The Inverse of matrix A is A^{-1} only if $A \times A^{-1} = I$ or $A^{-1} \times A = I$

Types of Matrices:

1. Row Matrix :

The Matrix which has only one row is called as row matrix

eg: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$

Note: Row matrix can have any number of columns.

2. Column Matrix :

A matrix which has only one column is called as column matrix

It can have any numbers of rows.

eg: $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

3. Square Matrix :

A matrix whose number of columns is equal to number of rows are called as square matrix.

eg:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

4. Null Matrix :

A matrix is said to be zero or null matrix if all the elements are zero.

eg: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. Diagonal Matrix:

A square matrix is called as diagonal matrix if its all non diagonal elements are zero

eg:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

→ Diagonal elements (1, 5, 9)
→ Non diagonal elements

6. Unit Matrix:

If a square matrix has diagonal elements as "1" and non diagonal elements as '0' is called unit matrix and It is denoted by "I".

eg: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. Upper Triangular Matrix:

A square matrix in which all elements below the diagonal are zero is called as upper Triangular Matrix.

eg:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

8. Lower Triangular Matrix:

A square matrix in which all elements above the diagonal are zero is called as lower Triangular Matrix.

eg:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

Rank of Matrix :

A non zero number 'r' is said to be rank of matrix A if

1). There exist atleast a number of A of order 'r'.

2). Every minor higher order than 'r' is zero.

The rank of matrix 'A' is denoted by $\rho(A) = r$.

The Rank of Matrix can be determine by :-

1. Minor Method.
2. Row Echelon form of Matrix.
3. Canonical form / Normal form of Matrix.

Number of non zero rows :

$$A = \begin{bmatrix} 1 & 5 & 8 & 10 \\ 0 & 2 & 6 & 9 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Rank = 4

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 7 & 2 & 4 & 1 \\ 14 & 4 & 8 & 2 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 7 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

Homogeneous Linear Equations :

An equation of the form $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is

called a homogenous equation.

$A - \lambda I$ is called Characteristic Matrix.
Here, A is any square matrix ; λ is a scalar and I is unit matrix.

The determinant $|A - \lambda I|$ is called the 'Characteristic polynomial of A'
The equation $|A - \lambda I| = 0$ is called as Characteristic equation of matrix A.

The roots of equation $|A - \lambda I|$ is called as Latent roots ; Characteristic roots ; or characteristic value or Eigen value or proper values of matrix A.

Suppose λ_1 is root of $|A - \lambda I| = 0$
then;

$$|A - \lambda_1 I| = 0$$

So further we find non zero column matrix X such that;

$$|A - \lambda_1 I| X = 0$$

The vector X is called as Eigen vector or Latent vector corresponding to the root λ_1.

Procedure to find Eigenvalues and Eigenvectors :

1. First write characteristic equation $|A - \lambda I| = 0$
2. Solve $|A - \lambda I|$ and find eigenvalues, λ_1; λ_2; λ_3.
3. For eigenvector put $\lambda = \lambda_1$ in $[A - \lambda_1 I] X = 0$
4. Now apply row transformation and reduce matrix $[A - \lambda_1 I]$
5. Now write equation and solve them to get eigenvector.

Eigenvalues

Q. Find eigenvalues of $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

Solution: The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\therefore \quad A - \lambda I = 0$$

$$= \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\therefore \quad = \begin{bmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix} = 0$$

$$(8-\lambda)[(3+\lambda)(\lambda-1)-8] + 8[4-4\lambda+6] - 2[-16+9+3\lambda] = 0$$

$$\therefore \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\therefore \quad \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$$

$$\therefore \quad (\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\therefore \quad (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\therefore \quad \lambda = 1;\ 2;\ 3$$

$$\lambda_1 = 1 \ ; \quad \lambda_2 = 2 \ ; \quad \lambda_3 = 3.$$

OR

The characteristic equation of A is $|A - \lambda I| = 0$

$$\lambda^3 - \left\{\text{sum of diagonal elements}\right\}\lambda^2 + \left\{\text{sum of minor of diagonal elements}\right\}\lambda - |A| = 0$$

$$\therefore \lambda^3 - \{8-3+1\}\lambda^2 + \left\{ \begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix} \right\}\lambda - \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + \{-11+14+8\}\lambda - 6 = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\therefore (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\therefore \lambda = 1, 2, 3$$

$$\therefore \lambda_1 = 1 \; ; \; \lambda_2 = 2 \; ; \; \lambda_3 = 3$$

The eigenvalues are same from both the methods.

Solved Example

Q. Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

$$\therefore \left| \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\therefore \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - \left(\text{Sum of diagonal elements}\right)\lambda^2 + \left(\text{sum of minor of diagonal elements}\right)\lambda - |A| = 0$$

$$\lambda^3 - (6+3+3)\lambda^2 + \left(\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} \right)\lambda - \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 0$$

$\therefore \lambda^3 - (12)\lambda^2 + [(9-1)+(18-4)+(18-4)]\lambda - 32 = 0$

$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\therefore (\lambda-2)(\lambda-8)(\lambda-2) = 0$

$\therefore \lambda = 8, 2, 2$

Eigenvalues are 8; 2; 2.

For $\lambda_1 = 8$

$[A - \lambda_1 I]X = 0$

$$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

forming equations

$$-2x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 - 5x_2 - x_3 = 0$$
$$2x_1 - x_2 - 5x_3 = 0$$

By crammer's rule

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x_1}{2-(-10)} = \frac{-x_2}{2-(-4)} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{-x_2}{6} = \frac{x_3}{6}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

hence corresponding to $\lambda = 8$ eigenvectors are $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 2$

$$[A - \lambda_2 I]X = 0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$ & $R_3 \to 2R_3 - R_1$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

forming equation

$$4x_1 - 2x_2 + 2x_3 = 0$$

i.e $\quad 2x_1 - x_2 + x_3 = 0$

Rank of matrix is '1'

Unknown is '3'

Hence, $n - r = 3 - 1 = 2$ linearly independant solution.

$$\therefore \quad 2x_1 = x_2 + (x_3)$$

$$\therefore \quad 2x_1 = x_2 - x_3$$

Put $x_2 = 0$ & $x_3 = 1$

$$2x_1 = 0 - 1$$

$$x_1 = \frac{-1}{2}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

Put $x_2 = 1$ and $x_3 = 0$

$$2x_1 = 1 - 0$$

$$x_1 = \frac{1}{2}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$$

Hence the corresponding to $\lambda = 2$ eigenvectors are $\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$; $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$

Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic equation.

Cayley Hamilton theorem states that square matrix (i.e A) can satisfy characteristic equation.

We can replace λ in characteristic equation with square matrix A.

Solved Example

Q. Verify Cayley Hamilton Theorem for the matrix A and hence find A^{-1}, A^{-2} and A^4

where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

prove that $A^{-1} = A^2 - 5A + 9I$

Solution: Step 1. The Characteristic Equation

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \begin{Bmatrix} \text{Sum of diagonal} \\ \text{Elements} \end{Bmatrix} \lambda^2 + \begin{Bmatrix} \text{Sum of minor of} \\ \text{Diagonal elements} \end{Bmatrix} \lambda + |A| = 0$$

$$\therefore \lambda^3 - \{1+3+1\}\lambda^2 + \left\{ \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \right\} \lambda + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = 0$$

$$\therefore \lambda^3 - 5\lambda^2 + \{(3-0)+(1-0)+(3+2)\}\lambda - 1 = 0$$

$\therefore \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$ is characteristic equation.

By Cayley Hamilton Theorem

$$A^3 - 5A^2 + 9A - I = 0 \quad \text{———— (1).}$$

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\therefore A^3 = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

$$A^3 - 5A^2 + 9A - I$$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence A statisfy characteristic equation, thus Cayley Hamilton theorem is verify.

1] A^{-1}

Multiply A^{-1} in eq^n(1)

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$\therefore A^{-1} = A^2 - 5A + 9I$$

$$A^{-1} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

ii) A^{-2}

Multiply A^{-1} by eqn (2)

$\therefore \quad A^{-1}.A^{-1} = A^{-1}.A^2 - 5A.A^{-1} + 9A^{-1}.I$

$\therefore \quad A^{-2} = A - 5I + 9A^{-1}$

$\therefore \quad A^{-2} = A + 9A^{-1} - 5I$

$$A^{-2} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 23 & 20 & 52 \\ 8 & 7 & 18 \\ 18 & -16 & 41 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} 23 & 20 & 52 \\ 8 & 7 & 18 \\ 18 & -16 & 41 \end{bmatrix}$$

iii) A^4

Multiply eqn (1) by A

$\therefore \quad A^4 - 5A^3 + 9A^2 - A = 0$

$\therefore \quad A^4 = 5A^3 - 9A^2 + A$

$$\therefore A^4 = 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$$

Similarity of Matrices:

If A and B are two square matrices of order n then B is said to be similar to A, if there exists a non singular matrix M such that

$$B = M^{-1}AM$$

A square matrix A is said to be diagonalisable if it is similar to a diagonal matrix.

Theorems

1). If A is similar to B, and B is similar to C, then A is similar to C.

2). If A and B are similar matrices then $|A| = |B|$.

3). If A is similar to B then A^2 is similar to B^2.

4). If A is diagonable then A^2 is diagonable.

5). If A and B are two similar matrices then they have the same eigenvalues.

Algebraic Multiplicity

If an eigenvalue λ_1 of matrix A is repeated t times then 't' is called algebraic multiplicity, of λ_1.

Geometric Multiplicity

If corresponding to an eigenvalue λ_1, there are s linearly independent eigenvectors then s is called geometric multiplicity of λ_1.

$$n - r = s$$

where n is no. of variables in matrix.
r is rank of the matrix.
s is independent eigenvectors of λ_1.

Diagonalising Matrix

A square non singular matrix A whose eigenvalues are all distinct can be diagonalised by similarity transformation,

$$D = M^{-1} AM$$

where M is matrix whose columns are eigenvectors of A and D is diagonal matrix whose diagonal elements are eigenvalues of A.

Note : i) If an eigenvalue of A is repeated then A may be diagonalisable or A may not be diagonalisable.

2) If Algebraic multiplicity and geometric multiplicity of repeated value are equal than A is diagonalisable.

3). If algebraic multiplicity and geometric multiplicity of repeated value are not equal than A is not diagonalisable.

Procedure :

1). Find eigenvalues of given matrix
2). If all eigenvalues are distinct, matrix is diagonalisable.
3). Find eigenvectors corresponding to eigenvalues.
4). Find diagonal matrix.
5). Use similarity transformation $D = M^{-1} AM$ and find M.

Eigenvalues are Repeated

1). Find eigenvalues.

2). If eigenvalues are repeated then,

3). Find Algebraic Multiplicity i.e t and Geometric multiplicity i.e s

4). If $t = s$ matrix is diagonalisable

5). If $t \neq s$ matrix is non diagonalisable

Solved Example

Q. Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalisable. Also find transforming matrix and diagonal matrix?

Solution : The characteristic equation of A is

$$\lambda^3 - (\text{sum of diagonal elements})\lambda^2 + (\text{minor of diagonal elements})\lambda - I = 0.$$

$$\therefore \lambda^3 - (8-3+1)\lambda^2 + (-16)\lambda - I = 0$$

$$\lambda^3 - 6\lambda^2 - 16\lambda - I = 0$$

$$\therefore \lambda = 1, 2, 3$$

$\because$ all eigenvalues of matrix A is distinct A is diagonalisable

for $\lambda_1 = 1$

$$[A - \lambda_1 I] X = 0$$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

forming equations

$$7x_1 - 8x_2 - 2x_3 = 0$$
$$4x_1 - 4x_2 - 2x_3 = 0$$
$$3x_1 - 4x_2 - 0x_3 = 0$$

By cramer's rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\therefore \frac{x_1}{8} = \frac{-x_2}{-6} = \frac{x_3}{4} = t$$

$\therefore x_1 = 8t$

$x_2 = 6t$

$x_3 = 4t$

$$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8t \\ 6t \\ 4t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

Eigenvector for $\lambda_1 = 1$ is

for $\lambda_2 = 2$ $[A - \lambda_2 I] X = 0$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

forming equations

$6x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 5x_2 - 2x_3 = 0$

$3x_1 - 4x_2 - x_3 = 0$

By crammer's rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{x_2}{4} = \frac{x_3}{2} = t$$

$\therefore x_1 = 6t$; $x_2 = 4t$; $x_3 = 2t$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \therefore X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ corresponding to $\lambda_2 = 2$ eigenvector is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

for $\lambda_3 = 3$ $\quad [A - \lambda_3 I]X = 0$

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ forming equations

$$5x_1 - 8x_2 - 2x_3 = 0$$
$$4x_1 - 6x_2 - 2x_3 = 0$$
$$3x_1 - 4x_2 - 2x_3 = 0$$

By Cramer's Rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{2} = t$$

$$\therefore \quad x_1 = 4t \; ; \; x_2 = 2t \; ; \; x_3 = 2t$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ corresponding to $\lambda_3 = 3$ eigenvector is $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Modal matrix $M = [X_1 \; X_2 \; X_3]$

$$X_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad X_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \quad M = [X_1 \ X_2 \ X_3]$

$$M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Since we know that

$$M^{-1}AM = D$$

where D is diagonal matrix.

$$\therefore \quad D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$\therefore \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Since,

$M^{-1}AM = D$ the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ will be

diagonalised to diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the

transforming matrix $M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Functions of a Square Matrix :

If A is a non singular square matrix with distinct eigenvalues we can find any power of A i.e A^K or A^n by :

$$A^n = MD^nM^{-1}$$

where M is modal matrix.

$$A^n = M \begin{bmatrix} \lambda_1^n & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n^n \end{bmatrix} M^{-1}$$

Solved Example

Q. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; find A^{50} ?

Solution : The characteristic equation of A

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$\Rightarrow (2-\lambda)^2 - 1 = 0$$

$$\therefore 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\therefore \lambda^2 - 4\lambda + 3 = 0$$

$$\therefore (\lambda-3)(\lambda-1) = 0$$

$$\therefore \lambda = 3 ; 1$$

for $\lambda_1 = 1$

$$[A - \lambda_1 I] X = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + x_2 = 0$

Putting $x_2 = -t$

$x_1 = t$

$$\therefore X_1 = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence eigenvector is [1, -1]

For $\lambda_2 = 3$

$[A - \lambda_2 I]X = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 + R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -x_1 + x_2 = 0$

$x_1 = x_2$

Putting $x_2 = t$; $x_1 = t$

$$\therefore X_2 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence; corresponding $\lambda = 3$ eigenvector is [1, 1]

$\therefore$ Modal Matrix $M = [X_1 \; X_2]$

$$\therefore M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$\because |M| = 2$

$$\therefore M^{-1} = \frac{1}{|M|} \cdot M^{-T}$$

$$M^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \left\{ \because D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \right\}$$

$$\therefore D^{50} = \begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix}$$

$$\therefore A^{50} = M D^{50} M^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 3^{50} \\ -1 & 3^{50} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore A^{50} = \frac{1}{2} \begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

Another Method :

If the matrix A is not diagonisable i.e they do not have distinct eigenvalues.

In such case we use another method

1). If matrix is of order 3x3

then;

$$\phi(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$$

where α_1, α_2, α_0 are constants to be determined.

2). If matrix is of order 2x2

then,

$$\phi(A) = \alpha_1 A + \alpha_0 I$$

where α_0, α_1 are constants to be determined.

for matrix 3X3 : $\phi(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$

for matrix 2X2 : $\phi(A) = \alpha_1 A + \alpha_0 I$

Note : we use this method when, diagonal matrix of A cannot be found out.

Solved Example

Q. If $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$ prove that $A^{50} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$

Solution : The characteristic equation of A is

$\because \begin{vmatrix} 2-\lambda & 3 \\ -3 & -4-\lambda \end{vmatrix}$

$\therefore (2-\lambda)(-4-\lambda) + 9 = 0$

$\therefore -8 - 2\lambda + 4\lambda + \lambda^2 + 9 = 0$

$\therefore \lambda^2 + 2\lambda + 1 = 0$

$\because (\lambda+1)^2 = 0$

$\therefore \lambda = -1; -1$

$\because$ eigenvalues are repeated we use second method.

For matrix of order 2x2

$\phi(A) = \alpha_1 A + \alpha_0 I$

Let $\phi(A) = A^{50} = \alpha_1 A + \alpha_0 I$

We assume

$\therefore \lambda^{50} = \alpha_1 \lambda + \alpha_0$ ——— (1)

putting $\lambda = -1$

$(-1)^{50} = \alpha_1(-1) + \alpha_0$

$\therefore 1 = -\alpha_1 + \alpha_0$ ——— (2)

differentiate eqn (1) by 'λ'.

$\therefore 50\lambda^{49} = \alpha_1$

Put $\lambda = -1$

$\therefore 50(-1)^{49} = \alpha_1$

$\therefore \alpha_1 = -50$

Substituting α_1 in eqn (2)

$\therefore 1 = -(-50) + \alpha_0$

$\therefore \alpha_0 = -49$

$\therefore$ from $A^{50} = \alpha_1 A + \alpha_0 I$

$$A^{50} = -50\begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -100 & -150 \\ 150 & 200 \end{bmatrix} + \begin{bmatrix} -49 & 0 \\ 0 & -49 \end{bmatrix}$$

$$\therefore A^{50} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$$

Hence proved.

Monic Polynomial of a Matrix

Let $f(x)$ be a polynomial in x and A be a square matrix of order n. if $f(A) = 0$ then we say that $f(x)$ annihilates the matrix A.

Caley Hamilton theorem states every matrix satisfies its Characteristic equation, thus characteristic polynomial of the matrix A annihilates A.

Monic Polynomial :

A polynomial in x in which the coefficient of the highest power of x is unity is called a monic polynomial.

eg : $x^3 - 2x^2 + 3x - 7$

Highest power of x is '3' and coefficient of highest power is unity hence its monic polynomial.

eg : $2x^3 - 3x^2 + 4x - 9$ is not monic polynomial, because Highest power of x is not unity

Minimal Polynomial of a matrix :

The monic polynomial of lowest degree that annihilates a matrix A is called minimal polynomial of A.

If $f(x)$ is minimal polynomial of A then equation $f(x)=0$ is called the minimal equation of the matrix A.

Note : If matrix is of order 'n'. Hence the degree of minimal polynomial of A cannot be greater than n.

Derogatory Matrices

If the degree of the minimal equation of a square matrix of order n is less than n, then it is called derogatory.

Non Derogatory Matrices:

If the degree of minimal equation of a square matrix of order n is equal to n, then it is called non derogatory.

Note :

1). Each eigenvalue of a square matrix is a root of the minimal equation.

2). If all eigenvalues are distinct then the matrix is called non derogatory matrix.

3). If all eigenvalues are distinct. Say $\lambda_1, \lambda_2, \lambda_3$ then, the minimal equation is

$$(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)=0$$

Steps to determine A is Derogatory or Non Derogatory.

1). Find characteristic eqⁿ of A and find eigenvalues.

2). If eigenvalues are distinct then matrix A is non derogatory.

3). If $\lambda_1 = \lambda_2$ then find

$$(x-\lambda_1)(x-\lambda_3) = 0 \text{ is satisfied by } A$$

the A is derogatory otherwise non derogatory.

4). If $\lambda_1 = \lambda_2 = \lambda_3$ then find $x - \lambda_1 = 0$ is satisfied by A then A is derogatory otherwise non derogatory.

Solved Example

Q. Show that $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory ?

Solution : Step 1: Characteristic equation of A

λ^3 – (sum of diagonal elements)λ^2 + (Sum of minors of diagonal elements)λ – $|A| = 0$

$$\lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0$$

$$\therefore (\lambda-3)(\lambda^2 - 15\lambda + 36) = 0$$

$$\therefore (\lambda-3)(\lambda-12)(\lambda-3) = 0$$

$\therefore$ eigenvalues are 3, 3, 12.

eigenvalues are also roots of minimal equation

Let $f(x)$ be minimal polynomial of A then,

$(x-3)(x-12)$ are factors of $f(x)$

$\therefore$ $x^2 - 15x + 36$ annihilates A or not

Find $A^2 - 15A + 36I = 0$

$$A^2 = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} \cdot \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix}$$

$\therefore A^2 - 15A + 36I$

$$= \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix} - 15\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore f(x) = x^2 - 15x + 36$ annihilates A.

Thus, f(x) is monic polynomial of lowest degree that annihilates A.

$\because$ degree is less than order of matrix A is derogatory.

Q. Show that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is non derogatory?

Solution: The characteristics equation of A is

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)[(3-\lambda)(5-\lambda)-16] - 2[2(5-\lambda)-12] + 3[8-3(3-\lambda)] = 0$

$\therefore (1-\lambda)[-1-8\lambda+\lambda^2] - 2[-2-2\lambda] + 3[-1+3\lambda] = 0$

$\therefore \lambda^3 - 9\lambda^2 - 6\lambda = 0$

$\therefore \lambda(\lambda^2 - 9\lambda - 6) = 0$

$\therefore$ all roots are distinct and since the characteristic equation is satisfied by A.

The degree of minimal equation is equal to 3 and hence A is non-derogatory.

Practice Problems

Q1. The sum of the eigenvalues of a 3x3 matrix is 6 and the product of the eigenvalue is also 6. If one of the eigenvalues is one, find the other two eigenvalues ?

Q2. Find the eigenvalues and eigenvector of the following matrices

a) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

Q3. Find the characteristic equation of the matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

Show that the matrix A satisfies the characteristic equation and hence find a) A^{-1} b) A^{-2} c) A^4

Q4. Find the characteristic equation of matrix A and verify that it satisfies Cayley Hamilton Theorem

Hence find A^{-1} and A^4

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Q5. Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalisable. Find the diagonal form D and the diagonalising matrix M ?

Q6. Find e^A and 4^A if $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

Q7. If $A = \begin{bmatrix} \pi/2 & \pi \\ 0 & 3\pi/2 \end{bmatrix}$ find $\sin A$?

Q8. Show that the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is derogatory ?

Q9. Show that $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is derogatory and find its minimal polynomial ?